NORTH-HOLLAND

On a Subclass of $\mathbf{P}_0$

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Dedicated to M. Fiedler and V. Pták on the occasion of their retirement.

Submitted by Wayne Barrett

ABSTRACT

In 1966, Fiedler and Pták wrote the first systematic investigation of the matrix class $\mathbf{P}_0$ consisting of all real square matrices with nonnegative principal minors. In this note, we focus on a particular subclass of $\mathbf{P}_0$ that has arisen within the field of linear complementarity theory, namely the class **SU** of sufficient matrices. Our principal result is that **SU** contains another class, $\mathbf{P}_*$. We conjecture that the latter two classes are in fact the same.

1. INTRODUCTION

When all the principal minors of a real square matrix are nonnegative, the matrix is said to belong to the class $\mathbf{P}_0$. This matrix class and its subclasses have been studied with some intensity for a period of at least thirty years, largely because of their prevalence in scientific computing (Cottle, Giannessi, and Lions [5], Harker and Pang [13]), complexity theory (Kojima, Megiddo, Noma, and Yoshise [16]), the theory of piecewise linear electrical networks (Bokhoven and Jess [3], Bokhoven [4], Katznelson [15], Sandberg and Willson

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[19]), and the theoretical foundations of the linear complementarity problem (LCP) (Aganagić and Cottle [1], Cottle, Pang, and Stone [7], Murty [17]). Much of this literature builds on the contributions of Fiedler and Pták [11], who were the first to study $\mathbf{P}_0$ in a systematic way. In that paper, Fiedler and Pták also considered the subclass $\mathbf{P}$ of matrices with positive principal minors, as others before them had done. (See, for example, Samelson, Thrall, and Wesler [18], Tucker [20], and Gale and Nikaido [12].) Earlier, the special cases of $\mathbf{P} \cap \mathbf{Z}$ and $\mathbf{P}_0 \cap \mathbf{Z}$ were also studied. (See Fiedler and Pták [10] and Berman and Plemmons [2] for a wealth of information on these topics.)

As suggested above, $\mathbf{P}_0$ and its subclasses play a prominent role in the theory of the linear complementarity problem (LCP). One of these subclasses is the class of *column sufficient* matrices first identified by Cottle, Pang, and Venkateswaran [8]. We denote this class by **CSU**. Recently, Kojima, Megiddo, Noma, and Yoshise [16] have enriched the literature of the LCP by introducing a nested family of subclasses of $\mathbf{P}_0$. The generic member of this family is denoted by $\mathbf{P}_*(\kappa)$, where κ is a nonnegative real number. The union of these over all $\kappa \geqslant 0$ is denoted by $\mathbf{P}_*$. Kojima et al. show that $\mathbf{P}_* \subset \mathbf{CSU}$. They also show that $\mathbf{P}_*$ is a subclass of $\mathbf{Q}_0$, another important matrix class pertaining to the existence of solutions to the LCP. What is peculiar about these results of Kojima et al. is that $\mathbf{P}_*$ is a subset of **CSU** and $\mathbf{P}_*$ is a subset of $\mathbf{Q}_0$, whereas **CSU** is not a subset of $\mathbf{Q}_0$. This presents a bit of a mystery. We address this by showing that $\mathbf{P}_* \subset \mathbf{RSU}$, the class of *row sufficient* matrices, which *is* a subset of $\mathbf{Q}_0$. This means that all $\mathbf{P}_*$-matrices belong to $\mathbf{CSU} \cap \mathbf{RSU} = \mathbf{SU}$, the class of *sufficient matrices*. We show that for matrices of order 2, the inclusion goes the other way. Whether this is true in general is presently an open question; we conjecture that the reverse inclusion holds for all orders.

2. (COLUMN AND ROW) SUFFICIENT MATRICES

For the reader's convenience, we state here the definitions of **CSU** and **RSU**. Next we provide a brief review of the definitions, importance, and properties of these classes. Full details are best seen in [8], [6], and [7].

DEFINITION. The matrix $M \in R^{n \times n}$ is *column sufficient* ($M \in \mathbf{CSU}$) if

$$x_i(Mx)_i \leqslant 0 \quad \text{for all } i \quad \Rightarrow \quad x_i(Mx)_i = 0 \quad \text{for all } i.$$

The matrix $M \in R^{n \times n}$ is *row sufficient* ($M \in \mathbf{RSU}$) if M^{T} is column sufficient, and M is *sufficient* ($M \in \mathbf{SU}$) if it is both row and column sufficient.

The classes **CSU** and **RSU** are not equal. This is easily seen from examples such as

$$M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

which are column (but not row) sufficient and row (but not column) sufficient, respectively. On the other hand, there are numerous examples of matrices that are row *and* column sufficient. For instance, all positive semidefinite matrices (regardless of symmetry) and all **P**-matrices are sufficient. So, obviously, is the direct sum of two matrices from these distinct matrix classes. Further examples of matrix classes contained in **SU** will be given below.

Interest in the classes **RSU** and **CSU** stems from the linear complementarity problem, that is, given $q \in R^n$ and $M \in R^{n \times n}$, find $x \in R^n$ which satisfies the conditions

(i) $x \geqslant 0$,
(ii) $q + Mx \geqslant 0$,
(iii) $x^{\mathrm{T}}(q + Mx) = 0$,

or show that no such x exists. An LCP with data q and M is denoted (q, M). A *solution* to (q, M) must satisfy (i), (ii), and (iii). A vector x is *feasible* for (q, M) if it satisfies (i) and (ii).

In studying the existence and multiplicity of solutions as well as algorithms for "processing" LCPs, researchers have found it helpful to exploit properties of various matrix classes. Indeed, some of these classes are defined in terms of properties of the LCP, while others are not so defined but have criteria for membership that are equivalent to properties of the LCP. This is illustrated by the classes **CSU** and **RSU**. The definitions of these two distinct matrix classes do not reveal their intimate connection with the LCP. In [8] it was proved that the solution set of the LCP (q, M) is convex for every q if and only if $M \in \mathbf{CSU}$. Furthermore, all minima of the quadratic program

$$\text{minimize} \quad x^{\mathrm{T}}q + x^{\mathrm{T}}Mx \qquad \text{subject to} \quad q + Mx \geqslant 0, \quad x \geqslant 0$$

are solutions of the LCP (q, M) for every q if and only if $M \in \mathbf{RSU}$.

In addition to **PSD** (the positive semidefinite matrices) and **P**, the class **SU** also contains the class **A** of *adequate* matrices (Ingleton [14]) and the class $\mathbf{P}_1$ of matrices M for which there is a unique index set β such that

$$\det M_{\beta\beta} = 0 \quad \text{and} \quad \det M_{\alpha\alpha} > 0 \quad \text{for all } \alpha \neq \beta .$$

(For details on $\mathbf{P}_1$ see Cottle and Stone [9].)

Column and row sufficient matrices have a number of significant properties. First, they are subclasses of $\mathbf{P}_0$, as can be seen from the definition and the characterization of $\mathbf{P}_0$ given by Fiedler and Pták [11]. Second, it is clear that they have the inheritance property,

$$M \in \mathbf{CSU} \quad \Rightarrow \quad M_{\alpha\alpha} \in \mathbf{CSU} \quad \text{for all } \alpha,$$
$$M \in \mathbf{RSU} \quad \Rightarrow \quad M_{\alpha\alpha} \in \mathbf{RSU} \quad \text{for all } \alpha.$$

(We use the term *complete* to describe a class with the inheritance property.) Third, these classes are invariant under *principal rearrangement*

$$M \in \mathbf{CSU} \quad \Rightarrow \quad P^TMP \in \mathbf{CSU} \quad \text{for all permutation matrices } P,$$
$$M \in \mathbf{RSU} \quad \Rightarrow \quad P^TMP \in \mathbf{RSU} \quad \text{for all permutation matrices } P;$$

they are also invariant under *principal pivotal transformation*

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix} \quad \rightarrow \quad \overline{M} = \begin{bmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1}M_{\alpha\beta} \\ M_{\beta\alpha}M_{\alpha\alpha}^{-1} & M_{\beta\beta} - M_{\beta\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\beta} \end{bmatrix}.$$

Here the principal submatrix $M_{\alpha\alpha}$ (called the *pivot block*) is assumed to be nonsingular, though it need not actually be a *leading* principal submatrix as shown above. Formally, we allow vacuous pivots ($\alpha = \varnothing$), in which case $\overline{M} = M$. Proofs of the invariance of **CSU** and **RSU** under principal pivoting are given as Theorem 4.1.7 and 4.1.8, respectively, in [7]. Finally, we mention that there are finite tests for column and row sufficiency [6]. One such result can be stated as follows.

LEMMA 2.1. *For $n \geqslant 2$, the matrix $M \in R^{n\times n}$ is column sufficient if and only if for every principal pivotal transform $\overline{M}$ of M*

1. $\overline{m}_{ii} \geqslant 0$ *for all* i;
2. *if* $\overline{m}_{ii} = 0$ *and* $\overline{m}_{ij} = 0$ $(j \neq i)$, *then* $\overline{m}_{ji} = 0$.

The two conditions of this lemma can be thought of as a criterion for a given square matrix of order 2 to be column sufficient. The lemma also says that $M \in R^{n \times n}$ is column sufficient if and only if every principal 2×2 submatrix of M and each of its principal pivotal transforms is column sufficient.

3. THE CLASSES $\mathbf{P}_*(\kappa)$ AND $\mathbf{P}_*$

Recently, Kojima, Megiddo, Noma, and Yoshise [16] introduced the matrix classes $\mathbf{P}_*(\kappa)$ and $\mathbf{P}_*$. The first of these is defined as follows. Let κ be any nonnegative real number. Then $\mathbf{P}_*(\kappa)$ consists all $n \times n$ matrices M satisfying

$$(1 + 4\kappa) \sum_{i \in I_+(x)} x_i(Mx)_i + \sum_{i \in I_-(x)} x_i(Mx)_i \geqslant 0 \qquad \text{for every} \quad x \in R^n, \tag{1}$$

where

$$I_+(x) = \{i : x_i(Mx)_i > 0\} \quad \text{and} \quad I_-(x) = \{i : x_i(Mx)_i < 0\}.$$

It is easy to see that $\mathbf{PSD} = \mathbf{P}_*(0)$, and $\mathbf{P}_*(\kappa)$ is isotone in the sense that $\mathbf{P}_*(\kappa_1) \subseteq \mathbf{P}_*(\kappa_2)$ if $0 \leqslant \kappa_1 \leqslant \kappa_2$. Define

$$\mathbf{P}_* = \bigcup_{\kappa \geqslant 0} \mathbf{P}_*(\kappa).$$

Kojima et al. [16] have proved that $\mathbf{P} \subset \mathbf{P}_* \subset \mathbf{CSU}$, the inclusions being proper. It is easy to see from the definition that, for each κ, the class $\mathbf{P}_*(\kappa)$ is complete.

Within this section we will discuss the set-theoretic relationship between $\mathbf{P}_*$ and $\mathbf{CSU}$. Our interest in understanding matrices of this class is related to several facts. Indeed, $\mathbf{P}_*$ (but not $\mathbf{CSU}$) is a subclass of $\mathbf{Q}_0$, the class of matrices M for which the LCP (q, M) has a solution whenever it is feasible [i.e., when there exists a feasible vector for (q, M)]. Moreover, the unified interior-point method of Kojima et al. [16] will solve the LCP (q, M) when it is feasible and $M \in \mathbf{P}_*$, but it will not always do so when $M \in \mathbf{CSU}$.

LEMMA 3.1. *For every $\kappa \geqslant 0$, the class $\mathbf{P}_*(\kappa)$ is invariant under principal pivoting. That is, if $M \in \mathbf{P}_*(\kappa)$ and* $\det M_{\alpha\alpha} \neq 0$, *then $\overline{M} \in$*

$\mathbf{P}_*(\kappa)$, *where* $\overline{M}$ *is the principal pivotal transform of* M *obtained by pivoting on* $M_{\alpha\alpha}$. *It follows that* $\mathbf{P}_*$ *is invariant under principal pivoting.*

Proof. Assume $M \in R^{n \times n}$, and let $u \in R^n$ be arbitrary. Define $v = \overline{M}u$, where $\overline{M}$ is obtained from M by pivoting on $M_{\alpha\alpha}$. If $i \in \alpha$, put $x_i = v_i$ and $y_i = u_i$. If $i \in \bar{\alpha}$, put $x_i = u_i$ and $y_i = v_i$. It then follows that $y = Mx$. Moreover,

$$\sum_{i \in I_+(u)} u_i v_i = \sum_{i \in I_+(x)} x_i y_i \quad \text{and} \quad \sum_{i \in I_-(u)} u_i v_i = \sum_{i \in I_-(x)} x_i y_i .$$

Thus, if $M \in \mathbf{P}_*(\kappa)$, it follows that $\overline{M} \in \mathbf{P}_*(\kappa)$. Since this holds for arbitrary $\kappa \geqslant 0$, the class $\mathbf{P}_*$ must also be invariant under principal pivoting. ∎

We now wish to describe $\mathbf{CSU} \setminus \mathbf{P}_*$ for the 2×2 case. Since **CSU** and $\mathbf{P}_*$ are both invariant under principal pivoting, and $\mathbf{P}_*$ contains **PSD** and **P**, it suffices to consider matrices M of the form

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} c & 0 \\ b & 0 \end{bmatrix}.$$

with further properties as stated below.

Case 1. If

$$M = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \quad \text{with} \quad bc < 0,$$

then $M \in \mathbf{CSU}$. In fact, a matrix of this form belongs to **PSD** if and only if $b + c = 0$, so we may rule this subcase out, if we wish. it follows from the definition that $M \in \mathbf{P}_*$. More specifically, $M \in \mathbf{P}_*(\kappa)$ where

$$4\kappa = \max\left\{ \frac{-(b+c)}{b}, \frac{-(b+c)}{c} \right\}.$$

Case 2. If

$$M = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \quad \text{and} \quad a > 0 > bc,$$

then $M \in \mathbf{CSU}$. A matrix of this form is **PSD** if and only if $b + c = 0$, so we may assume $b + c \neq 0$. It can be shown from the definition and a little analysis that $M \in \mathbf{P}_*$. More precisely, $M \in \mathbf{P}_*(\kappa)$, where

$$4\kappa = \max\left\{\frac{-(b+c)}{b}, \frac{-(b+c)}{c}\right\}.$$

To demonstrate this, we want to prove that

$$F(x) = \frac{-x^{\mathrm{T}}(Mx)}{\sum_{i \in I_+(x)} x_i (Mx)_i}$$

has a finite supremum; it suffices to seek this supremum over the set of x where $x^{\mathrm{T}}Mx < 0$. Since $F(\lambda x) = F(x)$ for all $\lambda \neq 0$, we can restrict attention to just two adjacent sides of the square

$$\{x \in R^2 : -1 \leqslant x_i \leqslant 1,\ i = 1, 2\}.$$

This sort of normalization happens to be particularly convenient. We first use the side where $x_1 = 1$ and then the side where $x_2 = 1$. To analyze F, we define

$$y_1 = ax_1 + bx_2,$$

$$y_2 = cx_1.$$

Then

$$x_1 y_1 = ax_1^2 + bx_1 x_2,$$

$$x_2 y_2 = cx_1 x_2,$$

and

$$x^{\mathrm{T}}y = x^{\mathrm{T}}(Mx) = ax_1^2 + (b + c)x_1 x_2.$$

There are four main subcases to consider, depending the signs of $b + c$ and c. For each sign pattern, we seek the supremum of F over the aforementioned sides intersected with the set of points where $x^{\mathrm{T}}Mx < 0$. In each subcase, it turns out that $F(x)$ is (or can be reduced to) the quotient of two linear forms with the denominator being positive over the corresponding line segment (domain). It is well known that finding the supremum of such a function over the line segment is a matter of evaluating the function at both end points and taking the larger value. In each case, we are led to the conclusion that

$$\sup\{F(x) : x^{\mathrm{T}}(Mx) < 0\} = \max\left\{\frac{-(b+c)}{b}, \frac{-(b+c)}{c}\right\}.$$

This completes the proof. (Interestingly, if $b + c = 0$, the formula still gives the right value for 4κ.)

Case 3. If

$$M = \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix} \quad \text{with} \quad c > 0, \quad b \neq 0,$$

then $M \in \mathbf{CSU}$. A matrix of this form cannot be positive semidefinite; hence there always exist vectors $x \in R^2$ such that $I_-(x) \neq \varnothing$. On the other hand, $2 \in I_+(x)$ for all $x \in R^2$ such that $x_2 \neq 0$. Let κ be an arbitrary positive scalar. In order for M to belong to the class $\mathbf{P}_*(\kappa)$ (and hence to $\mathbf{P}_*$), the inequality

$$\frac{-\left(bx_1x_2 + cx_2^2\right)}{cx_2^2} \leqslant 4\kappa$$

would have to be valid for all $x \in R^2$ such that $bx_1x_2 < 0$. Since this is not the case, it follows that $M \notin \mathbf{P}_*$.

Case 4. A matrix of the form

$$M = \begin{bmatrix} c & 0 \\ b & 0 \end{bmatrix} \quad \text{with} \quad c > 0, \quad b \neq 0$$

is a principal rearrangement of the matrix in case 3. Accordingly, it follows that $M \notin \mathbf{P}_*$.

We summarize this discussion in the following theorem.

THEOREM 3.2. *A 2×2 matrix M belongs to* $\mathbf{CSU} \setminus \mathbf{P}_*$ *if and only if it has the form*

$$\begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix} \text{ or } \begin{bmatrix} c & 0 \\ b & 0 \end{bmatrix}, \qquad \text{where} \quad b \neq 0 \text{ and } c > 0. \tag{2}$$

This characterization can be put another way.

THEOREM 3.3. *Let M be a 2×2 column sufficient matrix. Then $M \notin \mathbf{P}_*$ if and only if M is not row sufficient.*

Proof. By Theorem 3.2, $M \in \mathbf{CSU} \setminus \mathbf{P}_*$ if and only if it has the form (2). Such matrices are certainly not row sufficient. Conversely, if M belongs to $\mathbf{CSU} \setminus \mathbf{RSU}$, it cannot belong to $\mathbf{P}$ or $\mathbf{PSD}$, yet it must be a $\mathbf{P}_0$-matrix. This limits the possible forms to those indicated in (2). ■

Notice that this result implies

$$R^{2\times 2} \cap \mathbf{P}_* = R^{2\times 2} \cap \mathbf{SU}. \tag{3}$$

If (3) were valid for all orders n, then the classes $\mathbf{P}_*$ and $\mathbf{SU}$ would be identical.

THEOREM 3.4. *If $M \in \mathbf{P}_*$, then no principal pivotal transform of M has a 2×2 principal submatrix of the form in* (2).

Proof. This follows from Theorem 3.2, the completeness property of $\mathbf{P}_*$, and the invariance of this class under principal pivoting. ■

At present, we do not know if the converse of this theorem is true. If it were, we would have a finite test for membership in $\mathbf{CSU} \setminus \mathbf{P}_*$.

THEOREM 3.5. $\mathbf{P}_* \subseteq \mathbf{SU}$.

Proof. If M is a column sufficient matrix of order n, and M is not sufficient, then M is not a $\mathbf{P}_*$-matrix. This follows from Theorems 3.3 and 3.4 and the fact that M is a sufficient matrix if and only if every principal pivotal transform of M is sufficient of order 2. (A matrix M is said to be *sufficient of order* 2 if every 2×2 principal submatrix of M is sufficient. See [6].) It is now clear that $\mathbf{P}_* \subseteq \mathbf{SU}$. ■

REMARK. In general, column sufficient matrices do not always belong to $\mathbf{Q}_0$, yet from Theorem 3.3 and the discussion in [16, p. 39], it is known that $\mathbf{P}_* \subset \mathbf{Q}_0$. As stated earlier, the latter reference shows that $\mathbf{P}_* \subset \mathbf{CSU}$, but it does not include the observation that $\mathbf{P}_* \subseteq \mathbf{SU}$. It was shown by Cottle, Pang, and Venkateswaran [8] that $\mathbf{SU} \subset \mathbf{RSU} \subset \mathbf{Q}_0$. As a consequence, Theorem 3.5 provides an explanation for the enigma that $\mathbf{P}_*$-matrices belong to $\mathbf{Q}_0$.

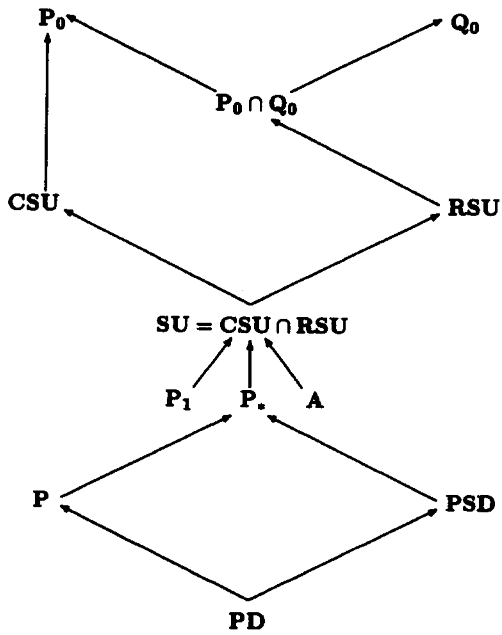

FIG. 1. Set-theoretic inclusions are indicated by upward arrows.

The set-theoretic inclusions discussed here are summarized in Figure 1. The notation **PD** in the figure stands for the class of positive definite matrices.

On the basis of (3) and some limited computational evidence with matrices of order greater than 2, we conjecture that $\mathbf{P}_* = \mathbf{SU}$.

Note Added in Proof: After the completion of this article, the conjecture that $\mathbf{P}_* = \mathbf{SU}$ was established by H. Väliaho. His paper $\mathbf{P}_*$ *matrices are just sufficient* has been accepted for publication in this journal.

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